Kiril Datchev
MA 495/595
Spring 2024

Homework 7

Due February 29th on paper at the beginning of class. Justify your answers. Please let me know if you have a question or find a mistake. There are some hints on the second page.

1. (a) Verify that the functions 1, x, and $\cos x$ are mutually orthogonal on $(-\pi, \pi)$.
   (b) Find constants a, b, and c that minimize $\int_{-\pi}^{\pi} |x^{37} - a - bx - c\cos x|^2 dx$.
2. Let D be the unit disk in $\mathbb{R}^2$, and consider the boundary value problem $\Delta u = 0$ in D, $u|_{\partial D} = f$, where $f = f(\theta)$ is a given continuous function on the unit circle.
   (a) In class we saw that if $r \in [0, 1)$ then the boundary value problem can be solved by

   $$u(r, \theta) = \sum_{n=\infty}^{\infty} a_n r^{|n|} e^{in\theta}, \qquad a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt,$$

   or equivalently by

   $$u(r, \theta) = \int_{-\pi}^{\pi} f(t) P_r(\theta - t) dt, \qquad P_r(\theta) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

   Prove that this solution is unique.
   (b) Prove that, for any $r \in [0, 1)$ and any θ, we have

   $$\min f \le u(r, \theta) \le \max f.$$

   What properties of P_r does this rely on?
   (c) Give the solution in the following cases, writing the final answer without any integrals or complex numbers:
      i. $f(\theta) = 1$,
      ii. $f(\theta) = \sin\theta$,
      iii. $f(\theta) = \sin^2\theta = \frac{1-\cos 2\theta}{2}$.
   (d) Rewrite the answers from part (d) in terms of x and y using $x = r\cos\theta$ and $y = r\sin\theta$. Recall that $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

Hints:

1. For part (a) use (7.8) and for (b) use orthogonal projection (7.19).
2. (a) Mimic the proof of Corollary 4.13, using the energy $E[u] = \int_D |\nabla u|^2$.

   (b) Plug $f(t) \leq \max f$ and $\min f \leq f(t)$ into the integral formula for u. The relevant properties of P can be extracted from this list, taken from Seeley's excellent *Introduction to Fourier Series and Integrals.*

   (i) $P(r, \phi)$ is even in ϕ, that is, $P(r, -\phi) = P(r, \phi)$.
   (ii) The maximum of $P(r, \phi)$ occurs at $\phi = 0$, and is $(1 + r)/2\pi(1 - r)$.
   (iii) $P(r, \phi)$ is monotone decreasing in $0 \leq \phi \leq \pi$, achieving a minimum of $(1 - r)/2\pi(1 + r)$ at $\phi = \pi$; thus $P > 0$.
   (iv) $\int_{-\pi}^{\pi} P(r, \phi)\, d\phi = 1$. (This is easily checked by integrating the series for $P(r, \phi)$.)

   (c) Use the series formula. Note that in each case the series has only a small number of nonzero terms.